CHARACTERIZATION OF THE SEQUENTIAL PRODUCT ON QUANTUM EFFECTS

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Abstract. We present a characterization of the standard sequential product of quantum effects. The characterization is in term of algebraic, continuity and duality conditions that can be physically motivated.

1. Introduction

This paper gives a set of five physically motivated conditions which fully characterize the sequential product on quantum effects. The positive operators on a complex Hilbert space $\mathcal{H}$ that are bounded above by the identity operator I are called the *quantum effects* on $\mathcal{H}$. The set of quantum effects on $\mathcal{H}$ is denoted by $\mathcal{E}(\mathcal{H})$. Quantum effects represent yes-no measurements that may be unsharp. The subset $\mathcal{P}(\mathcal{H})$ of $\mathcal{E}(\mathcal{H})$ consisting of orthogonal projections represent sharp yes-no measurements. Another important subset of $\mathcal{E}(\mathcal{H})$ is the set $\mathcal{D}(\mathcal{H})$ of density operators, i.e. the trace-class operators on $\mathcal{H}$ of unit trace, which represent the states of quantum systems. If $A \in \mathcal{E}(\mathcal{H})$ and $\rho \in \mathcal{D}(\mathcal{H})$ then $\mathrm{Tr}(\rho A)$ is the probability that A is observed (the answer is yes) when the system is in the state ρ.

A *sequential product* defined by $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ for any two quantum effects A, B has recently been introduced and studied [1, 5, 6, 7, 8, 9, 10]. The product $A \circ B$ represents the effect produced by first measuring A then measuring B. This product has also been generalized to an algebraic structure called a *sequential effect algebra (SEA)*. Examples of SEA are $[0,1] \subseteq \mathbb{R}$, Boolean algebras, fuzzy set systems $[0,1]^X$ and $\mathcal{E}(\mathcal{H})$. It has been shown that the sequential product is unique on all of these structures except $\mathcal{E}(\mathcal{H})$ and it has been an open problem whether $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ is the unique sequential product on $\mathcal{E}(\mathcal{H})$. It would be important physically to establish this uniqueness because

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we would then have an unambiguous form for the quantum mechanical sequential product.

There are various reasons for the appeal of the form $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ $(A, B \in \mathcal{E}(\mathcal{H}))$. First, when P and Q are orthogonal projections, then $P \circ Q = PQP$ is the accepted form for an ideal measurement in that case [2, 3, 4]. Second, $\circ$ satisfies various algebraic, continuity and duality conditions that one would expect from a sequential product. For example, for all $A, B, C \in \mathcal{E}(\mathcal{H})$ we have $I \circ A = A \circ I = A$, as well as $A \circ (B + C) = A \circ B + A \circ C$ whenever $B + C \in \mathcal{E}(\mathcal{H})$, $A \circ B \leq A$ and for all $\lambda \in [0, 1]$ we have $\lambda (A \circ B) = (\lambda A) \circ B = A \circ (\lambda B)$. Moreover, $\circ$ is jointly continuous for the strong operator topology. Finally, for any state $\rho \in \mathcal{D}(\mathcal{H})$ and quantum effects $A, B \in \mathcal{E}(\mathcal{H})$ we have the duality relation $\mathrm{Tr}(\rho(A \circ B)) = \mathrm{Tr}((A \circ \rho) B)$. We shall discuss the physical motivations for these conditions in the next section of this paper.

Our last reason for accepting the form $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ stems from quantum computation and information theory [11]. If $(A_i)_{i \in \mathbb{N}}$ is a sequence of bounded linear operators on $\mathcal{H}$ satisfying $\sum_{i=0}^{\infty} A_i^* A_i = I$ then the operators A_i $(i \in \mathbb{N})$ are called the *operational elements* of the *quantum operation* $\mathfrak{A} : \mathcal{D}(\mathcal{H}) \longrightarrow \mathcal{D}(\mathcal{H})$ defined by:

$$\mathfrak{A}(\rho) = \sum_{i=0}^{\infty} A_i \rho A_i^*. \tag{1.1}$$

Technically speaking, any trace preserving, normal, completely positive map has the form (1.1). Quantum operations are ubiquitous in quantum computation and information theory. They are used to describe dynamics, measurements, quantum channels, quantum interactions and quantum error correcting codes.

For a quantum measurement with outcomes labeled by $\mathbb{N}$, the operator $\mathfrak{A}(\rho)$ is the output state produced after the measurement is performed with input $\rho \in \mathcal{D}(\mathcal{H})$. If the outcome $i \in \mathbb{N}$ occurs, then an axiom of quantum mechanics says that the post measurement state becomes:

$$(\rho | A_i) = \frac{A_i \rho A_i^*}{\mathrm{Tr}(A_i \rho A_i^*)}. \tag{1.2}$$

Now, a very general type of measurement is called an *observable* and is modeled by a *positive operator-valued measure* (POVM). To keep this discussion simple, we shall only consider discrete observables. In this case, we can label the outcomes as before by $\mathbb{N}$ and the effect that the observable has outcome $i \in \mathbb{N}$ is denoted by $E_i \in \mathcal{E}(\mathcal{H})$. Since

one of the outcomes is always observed, we have $\sum_{i=0}^{\infty} E_i = I$. Therefore $\sum_{i=0}^{\infty} \left(E_i^{\frac{1}{2}}\right)^* \left(E_i^{\frac{1}{2}}\right) = \sum_{i=0}^{\infty} E_i = I$ so $\left(E_i^{\frac{1}{2}}\right)_{i\in\mathbb{N}}$ is the sequence of operational elements for the quantum operation $\mathfrak{A} : \rho \in \mathcal{D}(\mathcal{H}) \mapsto \sum_{i=0}^{\infty} E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}$, and (1.2) becomes:

$$\left(\rho | E_i^{\frac{1}{2}}\right) = \frac{E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}}{\operatorname{Tr}\left(E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}}\right)}. \tag{1.3}$$

Now, the real number $\mathbb{P}_{\rho}(E_i)$ defined by $\mathbb{P}_{\rho}(E_i) = \operatorname{Tr}(\rho E_i)$ is the probability that outcomes $i \in \mathbb{N}$ occurs in the state ρ and we can write (1.3) as:

$$E_i^{\frac{1}{2}} \rho E_i^{\frac{1}{2}} = \mathbb{P}_{\rho}(E_i)(\rho | E_i). \tag{1.4}$$

We can extend the quantum operation $\mathfrak{A}$ to $\mathcal{E}(\mathcal{H})$ and thus obtain, for all $F \in \mathcal{E}(\mathcal{H})$:

$$E_i^{\frac{1}{2}} F E_i^{\frac{1}{2}} = \mathbb{P}_{\rho}(E_i)(F | E_i). \tag{1.5}$$

Now, (1.5) is formally analogous to the formula for conditional probability in classical probability theory. In that case, it seems reasonable to interpret the left hand side of (1.5) as the formula for "E_i and F". However, $(F|E_i)$ is not symmetric in F and E_i but rather supposes that E_i was measured first. In the present noncommutative setting, we more precisely interpret $E^{\frac{1}{2}} F E^{\frac{1}{2}}$ $(E, F \in \mathcal{E}(\mathcal{H}))$ as the effect obtained from measuring E first and F second.

2. Physical Motivations

This section gives physical motivations for conditions that we shall use to characterize the sequential product on quantum effects. From now on in this paper, we shall always use $\circ$ to designate a general product on $\mathcal{E}(\mathcal{H})$ which satisfies the conditions given in this section. Later we shall establish that for all $A, B \in \mathcal{E}(\mathcal{H})$ we have $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ and thus that our conditions uniquely determine the sequential product on quantum effects.

A sequential product has two dual roles. When A and B are quantum effects, then $A \circ B$ is itself a quantum effect whose physical interpretation should be the effect measuring B after measuring A. On the other hand, given a state $\rho \in \mathcal{D}(\mathcal{H})$, since then $\rho \in \mathcal{E}(\mathcal{H})$ we can form $A \circ \rho$ for all $A \in \mathcal{E}(\mathcal{H})$. We shall impose on $\circ$ that the relation $\operatorname{Tr}(A \circ \rho) = \operatorname{Tr}(\rho A)$ must hold for all $A \in \mathcal{E}(\mathcal{H})$ — though in fact we will eventually retain a more general condition. In other words, $A \circ \rho$ is

a trace-class operator whose trace is the probability $\mathbb{P}_\rho(A)$ of observing A in ρ . From this, given any effect B, it is natural to interpret the probability $\mathrm{Tr}((A \circ \rho)\, B)$ as the probability to observe B and A in the state ρ, with the additional assumption that A is measured first. Let us assume now that $\mathbb{P}_\rho(A) = \mathrm{Tr}(A \circ \rho) \neq 0$. Then, it is natural to define the conditional probability of observing B given that A is observed first in the state ρ as the probability $\mathbb{P}_{\rho|A}(B)$ defined by:

$$\mathbb{P}_{\rho|A}(B) = \frac{\mathrm{Tr}((A \circ \rho)\, B)}{\mathrm{Tr}(A \circ \rho)}.$$

On the other hand, the probability of B given that A is observed first, computed in the original state ρ, should be given by:

$$\mathbb{P}_\rho(B|A) = \frac{\mathrm{Tr}(\rho(A \circ B))}{\mathrm{Tr}\,(A \circ \rho)}$$

since $A \circ B$ precisely represents the effect of observing B after A. It appears reasonable to impose on $\circ$ that both these probabilities should be equal as they should describe the same event. Thus, if $\mathrm{Tr}(A \circ \rho) \neq 0$ we should have $\mathbb{P}_{\rho|A}(B) = \mathbb{P}_\rho(B|A)$. Simplifying by $\mathrm{Tr}(A \circ \rho)$ and generalizing to all of $\mathcal{E}\,(\mathcal{H})$ gives us:

Condition 1. (Duality) *A sequential product $\circ$ satisfies the relation:*

$$\mathrm{Tr}((A \circ \rho)\, B) = \mathrm{Tr}(\rho(A \circ B))$$

for all states $\rho \in \mathcal{D}\,(\mathcal{H})$ and all quantum effects $A, B \in \mathcal{E}\,(\mathcal{H})$.

Note that since $\rho \in \mathcal{D}\,(\mathcal{H})$ is trace-class, so is $\rho(A \circ I)$ and thus Condition (1) implies that $\mathrm{Tr}\,(A \circ \rho) = \mathrm{Tr}\,(\rho(A \circ I))$ so $A \circ \rho$ is trace-class as well, of trace in $[0, 1]$. Now, Condition (1) implies that $\circ$ must be affine in its second variable. It will be useful to record this fact for our discussion:

Lemma 2.1. *Let us assume that $\circ$ satisfies Condition (1). Then for all $A, B, C \in \mathcal{E}\,(\mathcal{H})$ and all $\lambda \in [0, 1]$ we have:*

$$A \circ (\lambda B + (1 - \lambda)\, C) = \lambda\, (A \circ B) + (1 - \lambda)\, (A \circ C)$$

i.e. $B \mapsto A \circ B$ is affine on the convex set $\mathcal{E}\,(\,\mathcal{H})$.

In particular, if η is a trace class operator on $\mathcal{H}$ with trace $\lambda \in [0, 1]$ then for all $A, B \in \mathcal{E}\,(\mathcal{H})$ we have:

$$\mathrm{Tr}\,((A \circ \eta)\, B) = \mathrm{Tr}\,(\eta\, (A \circ B))\,.$$

Proof. Let $A,B,C\in\mathcal{E}(\mathcal{H})$ and $\rho\in\mathcal{D}(\mathcal{H})$ for all this proof.

Let $\lambda\in[0,1]$. Then:

$$\begin{aligned}\operatorname{Tr}(\rho(A\circ(\lambda B))) &= \operatorname{Tr}((A\circ\rho)(\lambda B)) \text{ by Condition (1),}\\ &= \lambda\operatorname{Tr}((A\circ\rho)B)\\ &= \lambda\operatorname{Tr}(\rho(A\circ B)) \text{ by Condition (1) again.}\end{aligned}$$

Since ρ is arbitrary, we deduce that $A\circ(\lambda B)=\lambda(A\circ B)$.

Thus, let η be a trace-class operator of trace $\lambda\in(0,1]$. Then:

$$\begin{aligned}\operatorname{Tr}(\eta(A\circ B)) &= \lambda\operatorname{Tr}\left(\frac{1}{\lambda}\eta(A\circ B)\right)\\ &= \lambda\operatorname{Tr}\left(\left(A\circ\left(\frac{1}{\lambda}\eta\right)\right)B\right) \text{ by Condition (1),}\\ &= \operatorname{Tr}((A\circ\eta)B) \text{ by our work above.}\end{aligned}$$

We prove additivity in a similar manner. We have:

$$\begin{aligned}\operatorname{Tr}(\rho(A\circ(B+C))) &= \operatorname{Tr}((A\circ\rho)(B+C)) \text{ by Condition (1),}\\ &= \operatorname{Tr}((A\circ\rho)B)+\operatorname{Tr}((A\circ\rho)C)\\ &= \operatorname{Tr}(\rho(A\circ B+A\circ C))\end{aligned}$$

where we used Condition (1) again. Once again, since ρ is an arbitrary state, we conclude that $A\circ(B+C)=A\circ B+A\circ C$. □

The identity I of $\mathcal{H}$ is the effect which always measures 1, or yes, no matter what state the quantum system is in. Consequently, measuring I does not affect the quantum system (which reflects the fact that I commutes with all operators). So measuring I before or after measuring $A\in\mathcal{E}(\mathcal{H})$ should not change the simple measurement of A. Formally, we shall henceforth assume that:

Condition 2. (Unit) *A sequential product $\circ$ needs to satisfy:*

$$A\circ I=I\circ A=A$$

for all $A\in\mathcal{E}(\mathcal{H})$.

We note that given $\rho\in\mathcal{E}(\mathcal{H})$ and $A,B\in\mathcal{E}(\mathcal{H})$ we have:

$$\begin{aligned}\operatorname{Tr}((A\circ\rho)B) &= \operatorname{Tr}((A\circ\rho)(B\circ I)) \text{ by Condition (2),}\\ &= \operatorname{Tr}(B\circ(A\circ\rho)) \text{ by Lemma (2.1),}\end{aligned} \tag{2.1}$$

since $A\circ\rho$ is trace-class of trace in $[0,1]$.

More generally, suppose we are given two quantum effects A, B. Let us assume that A and B commute. Physically, we are therefore assuming that measurements of A do not affect B and vice-versa. Therefore, the sequential product should be symmetric: measuring A first and B second should be the same as measuring B first and A second. Even more concretely, since A and B commute, they can be measure simultaneously and this measurement is given by the effect AB. Thus, we can physically expect that $A \circ B = AB = BA = B \circ A$. We actually will only require a special case of this observation: namely, that $A^2 = A \circ A$ for all $A \in \mathcal{E}(\mathcal{H})$.

Let us generalize this principle further. Let $\rho \in \mathcal{D}(\mathcal{H})$ be a state of a quantum system. Let A, B be two quantum effects. We can view A and B as two successive unsharp filters. We can proceed with a first experiment by sending the state ρ through A and measure the resulting state as $\frac{1}{\mathrm{Tr}(A\circ\rho)}(A \circ \rho)$. Thus our quantum system is in a new state, and can be sent through the second filter B. Measuring the state at the exit of B we shall see the state $\frac{1}{\mathrm{Tr}(B\circ(A\circ\rho))}(B \circ (A \circ \rho))$.

Alternatively, we may send the system in its state ρ through the compound filter $A \circ B$ which performs first A then B and measure the resulting state at once. We then would get $\frac{1}{\mathrm{Tr}((A\circ B)\circ\rho)}((A \circ B) \circ \rho)$. In general, these two experiments lead to different states. However, the normalizations are the same:

$$
\begin{aligned}
\mathrm{Tr}\left((A \circ B) \circ \rho\right) &= \mathrm{Tr}\left(\rho\left((A \circ B) \circ I\right)\right) \text{ by Condition (1),} \\
&= \mathrm{Tr}\left(\rho\left(A \circ B\right)\right) \text{ by Condition (2),} \\
&= \mathrm{Tr}\left(\left(A \circ \rho\right) B\right) \text{ by Condition (1),} \\
&= \mathrm{Tr}\left(\left(B \circ \left(A \circ \rho\right)\right)\right) \text{ by Equality (2.1).}
\end{aligned}
$$

Let us now assume that A and B commute. We have seen already that we expect $AB = A \circ B$ and thus the compound filter has no "internal side effects". Thus it should make no difference which of the two experiments we conduct: we ought to obtain the same output state from the input ρ. Hence, we obtain that for all states ρ, if A and B commute then:

$$
B \circ (A \circ \rho) = (A \circ B) \circ \rho. \tag{2.2}
$$

Using our duality assumption, we can deduce that for any effect $C \in \mathcal{E}(\mathcal{H})$ we have, by successive applications of Condition (1):

$$\begin{aligned} \operatorname{Tr}(\rho((A \circ B) \circ C)) &= \operatorname{Tr}(((A \circ B) \circ \rho) C) \\ &= \operatorname{Tr}((B \circ (A \circ \rho)) C) \text{ by Equality (2.2),} \\ &= \operatorname{Tr}((A \circ \rho)(B \circ C)) \\ &= \operatorname{Tr}(\rho(A \circ (B \circ C))) . \end{aligned}$$

As this is valid for all ρ we conclude that if $AB = BA$ then $(A \circ B) \circ C = A \circ (B \circ C)$. In fact, we shall only require a special case of this relation, together with the observation that $A^2 = A \circ A$. We thus state:

Condition 3. (Weak associativity) *A sequential product $\circ$ needs to satisfy the relation:*

$$A \circ (A \circ B) = (A \circ A) \circ B = A^2 \circ B$$

for all $A, B \in \mathcal{E}(\mathcal{H})$.

We shall require two more properties of a sequential product. First of all, we desire the sequential product to be continuous. We saw that any sequential product will be convex in its second variable which, with a little work and our other assumptions, will grant continuity in the second variable automatically. However, we also wish some form of continuity on the first variable. We state:

Condition 4. (Continuity) *Let $B \in \mathcal{E}(\mathcal{H})$ be given. Then*

$$A \in \mathcal{E}(\mathcal{H}) \mapsto A \circ B$$

is continuous in the strong operator topology.

The last condition which we impose on any sequential product is preservation of pure states (up to normalization). A vector state, or a pure state, is a rank-one orthogonal projection. Thus, let ρ be a pure state. If $A \circ \rho \neq 0$ for $A \in \mathcal{E}(\mathcal{H})$ then it is reasonable that the state $\rho | A = \frac{1}{\operatorname{Tr}(\rho A)} (A \circ \rho)$ conditioned on observing A should again be pure.

Condition 5. (Purity) *Let p be a rank one orthogonal projection. Then for all $A \in \mathcal{E}(\mathcal{H})$ the effect $A \circ p$ is of rank 1 or 0.*

3. The Characterization Theorem

We define a sequential product on $\mathcal{E}(\mathcal{H})$ by incorporating Conditions 1-5 from the previous section:

Definition 3.1. *A sequential product* $\circ$ *on* $\mathcal{E}(\mathcal{H})$ *is a binary operation on* $\mathcal{E}(\mathcal{H})$ *satisfying Conditions 1-5, namely: for all* $A, B \in \mathcal{E}(\mathcal{H})$*:*

(1) *For all* $\rho \in \mathcal{D}(\mathcal{H})$ *we have:*
$$\operatorname{Tr}((A \circ \rho) B) = \operatorname{Tr}(\rho(A \circ B)),$$
(2) *We have* $A \circ I = I \circ A = A$,
(3) *We have* $A^2 \circ B = A \circ (A \circ B)$,
(4) *The map* $E \in \mathcal{E}(\mathcal{H}) \mapsto E \circ B$ *is continuous in the strong topology,*
(5) *If* P *is a pure state then* $\frac{1}{\operatorname{Tr}(A \circ P)}(A \circ P)$ *is a pure state whenever* $A \circ \rho \neq 0$.

We check trivially that:

Proposition 3.2. *The product defined by* $A, B \in \mathcal{E}(\mathcal{H}) \mapsto A^{\frac{1}{2}} B A^{\frac{1}{2}}$ *is a sequential product on* $\mathcal{E}(\mathcal{H})$.

We shall now prove the converse of the Proposition (3.2). We shall use the following notations. The set of all trace-class operators on $\mathcal{H}$ is denoted by $\mathcal{T}(\mathcal{H})$. The set of positive trace-class operators is denoted by $\mathcal{T}^+(\mathcal{H})$. An element $A \in \mathcal{T}^+(\mathcal{H})$ is *pure* if whenever $0 \leq B \leq A$ there exists $\lambda \in [0,1]$ such that $B = \lambda A$. Clearly, pure elements are of the form λP for $\lambda \in [0,1]$ and P a rank-one projection. A linear map $T : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$ is *positive* when $T(\mathcal{T}^+(\mathcal{H})) \subseteq \mathcal{T}^+(\mathcal{H})$ and is pure when $T(A)$ is pure for all pure elements A of $\mathcal{T}(\mathcal{H})$.

We now have:

Theorem 3.3. *A map* $\circ : \mathcal{E}(\mathcal{H}) \times \mathcal{E}(\mathcal{H}) \longrightarrow \mathcal{E}(\mathcal{H})$ *is a sequential product on* $\mathcal{E}(\mathcal{H})$ *if and only if for all* $A, B \in \mathcal{E}(\mathcal{H})$ *we have* $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$.

Proof. The sufficient condition is Proposition (3.2). Let us now prove that the condition is necessary as well.

Let $\circ$ be a map satisfying Conditions 1-5. For $A \in \mathcal{E}(\mathcal{H})$, we set $\Phi_A : B \in \mathcal{E}(\mathcal{H}) \mapsto A \circ B$. If $\rho \in \mathcal{D}(\mathcal{H})$ then:
$$\operatorname{Tr}(A \circ \rho) = \operatorname{Tr}(\rho(A \circ I)) = \operatorname{Tr}(\rho A)$$
so $\Phi_A(\rho) \in \mathcal{T}^+(\mathcal{H})$. By Lemma (2.1), the map Φ_A is affine on the convex set $\mathcal{E}(\mathcal{H})$. Since $\mathcal{E}(\mathcal{H})$ generates algebraically the vector space

$\mathcal{B}(\mathcal{H})$ of all bounded linear operators on $\mathcal{H}$, it follows that Φ_A has a unique linear extension, which we also denote by Φ_A , to $\mathcal{B}(\mathcal{H})$. Since $\mathrm{Tr}(\Phi_A(\rho)) = \mathrm{Tr}(\rho A) \leq 1$ for all $\rho \in \mathcal{D}(\mathcal{H})$ we conclude that the restriction of Φ_A to $\mathcal{T}(\mathcal{H})$ is a pure positive linear map from $\mathcal{T}(\mathcal{H})$ to $\mathcal{T}(\mathcal{H})$. It follows from [4, Theorem 3.1] that $\Phi_A : \mathcal{T}(\mathcal{H}) \longrightarrow \mathcal{T}(\mathcal{H})$ has one of the following forms:

i) There exists $C \in \mathcal{B}(\mathcal{H})$ such that for all $\rho \in \mathcal{T}(\mathcal{H})$ we have $\Phi_A(\rho) = C^*\rho C$,

ii) There exists a bounded conjugate linear map C on $\mathcal{H}$ such that for all $\rho \in \mathcal{T}(\mathcal{H})$ we have $\Phi_A(\rho) = C^*\rho^* C$,

iii) There exists $B \in \mathcal{B}(\mathcal{H})^+$ and some orthogonal projection P_ψ on the span of some unit vector ψ such that for all $\rho \in \mathcal{T}(\mathcal{H})$ we have $\Phi_A(\rho) = \mathrm{Tr}(\rho B) P_\psi$.

We first deal with case (iii). In this case:

$$\mathrm{Tr}(\omega\Phi_A(\rho)) = \mathrm{Tr}(\rho B)\langle\omega\psi,\psi\rangle \tag{3.1}$$

for all $\omega \in \mathcal{T}^+(\mathcal{H})$. Let $(\omega_i)_{i\in\Lambda}$ be an increasing net in $\mathcal{B}(\mathcal{H})$ which converges to I in the strong operator topology.

Applying (3.1) we have:

$$\begin{aligned}\mathrm{Tr}(\rho A) &= \mathrm{Tr}(\Phi_A(\rho)) = \lim_{i\in\Lambda}\mathrm{Tr}(\omega_i\Phi_A(\rho))\\ &= \mathrm{Tr}(\rho B)\lim_{i\in\Lambda}\langle\omega_i\psi,\psi\rangle = \mathrm{Tr}(\rho B)\end{aligned}$$

for every $\rho \in \mathcal{T}(\mathcal{H})$. Hence $B = A$ and we have:

$$A \circ \rho = \mathrm{Tr}(\rho A) P_\psi \tag{3.2}$$

for all $\rho \in \mathcal{T}(\mathcal{H})$. Applying (3.2) and Condition (1) we conclude that:

$$\begin{aligned}\mathrm{Tr}(\rho A)\langle\omega\psi,\psi\rangle &= \mathrm{Tr}(\omega(A\circ\rho))\\ &= \mathrm{Tr}((\omega\circ A)\rho)\\ &= \mathrm{Tr}(\omega A)\langle\rho\psi,\psi\rangle \qquad (3.3)\end{aligned}$$

for all $\rho, \omega \in \mathcal{D}(\mathcal{H})$. In (3.3), let $\rho = P_\varphi$ for some unit vector $\varphi \in \mathcal{H}$ with $\langle\psi,\varphi\rangle = 0$. Then $\mathrm{Tr}(P_\varphi A)\langle\omega\psi,\psi\rangle = 0$ for all $\omega \in \mathcal{D}(\mathcal{H})$. Hence, $\langle A\varphi,\varphi\rangle = 0$ so $A\varphi = 0$ since $A \geq 0$. It follows that $A = \lambda P_\psi$ for some $\lambda \in [0,1]$. By (3.2) we have:

$$\begin{aligned}A\circ\rho &= \lambda\,\mathrm{Tr}(\rho P_\psi)P_\psi = \lambda\langle\rho\psi,\psi\rangle P_\psi\\ &= \sqrt[2]{\lambda}P_\psi\rho\sqrt[2]{\lambda}P_\psi = A^{\frac{1}{2}}\rho A^{\frac{1}{2}}. \qquad (3.4)\end{aligned}$$

We now show that the map $B \mapsto A \circ B$ is normal. First, notice that if $B, C \in \mathcal{E}(\mathcal{H})$ and $B \leq C$ then $C - B \in \mathcal{E}(\mathcal{H})$ and we have:

$$A\circ C = A\circ[B + (C - B)] = A\circ B + A\circ(C - B) \geq A\circ B.$$

Next suppose that $(B_i)_{i\in\Lambda}$ is an increasing net converging to B in the strong operator topology. By Condition (1) we have:

$$\begin{aligned}\lim_{i\in\Lambda}\operatorname{Tr}\left(\rho\left(A\circ B_i\right)\right) &= \lim_{i\in\Lambda}\operatorname{Tr}\left(\left(A\circ\rho\right)B_i\right)=\operatorname{Tr}\left(\left(A\circ\rho\right)B\right)\\ &= \operatorname{Tr}\left(\rho\left(A\circ B\right)\right).\end{aligned}$$

Hence $(A\circ B_i)_{i\in\Lambda}$ converges to $A\circ B$ in the ultraweak topology. Since $(A\circ B_i)_{i\in\Lambda}$ is an increasing net, it converges strongly to $A\circ B$ [4, 12]. To complete case (iii), for $B\in\mathcal{E}(\mathcal{H})$ there exists an increasing net $(\rho_i)_{i\in\Lambda}$ in $\mathcal{D}(\mathcal{H})$ converging strongly to B. Applying (3.4) and the normality of $B\mapsto A\circ B$ we conclude that:

$$A\circ B=\lim_{i\in\Lambda}A\circ\rho_i=\lim_{i\in\Lambda}A^{\frac{1}{2}}\rho_iA^{\frac{1}{2}}=A^{\frac{1}{2}}BA^{\frac{1}{2}}.$$

We now treat case (i) and omit case (ii) which is dealt with in a similar manner as (i). By normality we therefore can assume that there exists $C\in\mathcal{B}(\mathcal{H})$ such that for all $B\in\mathcal{E}(\mathcal{H})$ we have $A\circ B=C^*BC$. Now

$$A=A\circ I=C^*C=|C|^2$$

so $|C|=A^{\frac{1}{2}}$. By the Polar Decomposition Theorem, there exists a partial isometry U on $\mathcal{H}$ such that $C=UA^{\frac{1}{2}}$. Then

$$A=A^{\frac{1}{2}}U^*UA^{\frac{1}{2}}. \tag{3.5}$$

We now assume that A is invertible. It follows from (3.5) that $U^*U=1$. Applying Condition (1) gives:

$$\begin{aligned}\operatorname{Tr}\left(A^{\frac{1}{2}}U^*BUA^{\frac{1}{2}}\rho\right) &= \operatorname{Tr}\left(BA^{\frac{1}{2}}U^*\rho UA^{\frac{1}{2}}\right)\\ &= \operatorname{Tr}\left(UA^{\frac{1}{2}}BA^{\frac{1}{2}}U^*\rho\right)\end{aligned}$$

for every $\rho\in\mathcal{D}(\mathcal{H})$. It follows that:

$$A^{\frac{1}{2}}U^*BUA^{\frac{1}{2}}=UA^{\frac{1}{2}}BA^{\frac{1}{2}}U^* \tag{3.6}$$

for every $B\in\mathcal{E}(\mathcal{H})$. In particular, with $B=I$ we have $A=UAU^*$ and since $U^*U=I$ we have $UA=AU$. Hence $A=AUU^*$ and since A is invertible, we have $UU^*=I$. It follows that U is unitary. Moreover, from (3.6) we have that:

$$A^{\frac{1}{2}}U^*BUA^{\frac{1}{2}}=A^{\frac{1}{2}}UBU^*A^{\frac{1}{2}}$$

and using the invertibility of A again, we obtain that $U^*BU=UBU^*$ for every $B\in\mathcal{E}(\mathcal{H})$. It follows that $U^2B=BU^2$ for all $B\in\mathcal{E}(\mathcal{H})$ so $U^2=\mu I$ with $\mu\in\mathbb{C}$ such that $|\mu|=1$.

We now apply Condition (3) and obtain:

$$\begin{aligned} A^2 \circ B &= A \circ (A \circ B) = A \circ \left(A^{\frac{1}{2}} U^* B U A^{\frac{1}{2}}\right) \\ &= A^{\frac{1}{2}} U^* A^{\frac{1}{2}} U^* B U A^{\frac{1}{2}} U A^{\frac{1}{2}} \\ &= U^{*2} A B A U^2 = ABA. \end{aligned}$$

Replacing A by $A^{\frac{1}{2}}$ we thus get $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ for all $B \in \mathcal{E}(\mathcal{H})$.

Now, let $A \in \mathcal{E}(\mathcal{H})$ not invertible. Then for all $i \in \mathbb{N}\setminus\{0\}$ we set $A_i = \left(1 + \frac{1}{i}\right)^{-1}\left(A + \frac{1}{i} I\right)$ and note that $A_i \in \mathcal{E}(\mathcal{H})$ is invertible. The sequence $(A_i)_{i \in \mathbb{N}\setminus\{0\}}$ converges strongly to A. It follows from Condition (4) that $A \circ B = A^{\frac{1}{2}} B A^{\frac{1}{2}}$ for all $B \in \mathcal{E}(\mathcal{H})$. □

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